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Application of Undetermined Coefficients Method in Solving Fractional Integral Problems

Chii-Huei Yu

School of Mathematics and Statistics,

Zhaoqing University, Guangdong, China

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***Abstract*: In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study two fractional integral problems. The solutions of these two fractional integrals can be obtained by using undetermined coefficients method. In fact, our results are generalizations of the ordinary calculus results.**

***Keywords*: Jumarie's modified R-L fractional calculus, new multiplication, fractional analytic functions, fractional integrals, undetermined coefficients method.**

I. INTRODUCTION

In the second half of the 20th century, a considerable number of studies on fractional calculus were published in the engineering literature. In fact, fractional calculus has many applications in physics, mechanics, viscoelasticity, economics, mathematical biology, electrical engineering, control theory, and other fields [1-14]. However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [15-19]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, the following two fractional integral problems are studied:

$$\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+2\right)\otimes_{\alpha}\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\otimes_{\alpha}\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\right]^{\otimes_{\alpha}(-1)}\right],$$

and

$$\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-3\right)\otimes_{\alpha}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\otimes_{\alpha}\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}2}\right)\right]^{\otimes_{\alpha}(-1)}\right].$$

Where $0<\alpha\leq 1$. Using undetermined coefficients method, the solutions of these two fractional integrals can be obtained. Moreover, our results are generalizations of the results of classical calculus.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([20])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([21])**:** *If α, β, x_0, c are real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([22])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([23])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\ . \end{aligned} \tag{8}$$

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Definition 2.5 ([24])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\big(g_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\big(f_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([25])**:** Let $0<\alpha\leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha\circ g_\alpha)(x^\alpha)=(g_\alpha\circ f_\alpha)(x^\alpha)=\frac{1}{\Gamma(\alpha+1)}x^\alpha. \tag{13}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([26])**:** If $0<\alpha\leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

Definition 2.8 ([27])**:** Let $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}=f_\alpha(x^\alpha)\otimes_\alpha\cdots\otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha(-1)}$.

III. EXAMPLES

In this section, we will give some examples to illustrate how to use undetermined coefficients method to solve fractional integral problems.

Example 3.1: Let $0<\alpha\leq 1$. Find the α-fractional integral

$$\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+2\right)\otimes_\alpha\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right)\otimes_\alpha\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right)\right]^{\otimes_\alpha(-1)}\right]. \tag{15}$$

Solution By undetermined coefficients method, let

$$\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+2\right)\otimes_\alpha\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right)\otimes_\alpha\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right)\right]^{\otimes_\alpha(-1)}$$

$$=A\cdot\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right)\right]^{\otimes_\alpha(-1)}+\left(B\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+C\right)\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right]^{\otimes_\alpha(-1)}. \tag{16}$$

Where A,B,C are constants. Then

$$\frac{1}{\Gamma(\alpha+1)}x^\alpha+2=A\cdot\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right]+\left(B\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+C\right)\otimes_\alpha\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right). \tag{17}$$

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Therefore,

$$\begin{cases} A+2B=0, \\ A+B+2C=1, \\ A+C=2. \end{cases} \tag{18}$$

And hence,

$$\begin{cases} A=2, \\ B=-1, \\ C=0. \end{cases} \tag{19}$$

Thus,

$$\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+2\right)\otimes_{\alpha}\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\otimes_{\alpha}\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\right]^{\otimes_{\alpha}(-1)}\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[2\cdot\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\right]^{\otimes_{\alpha}(-1)}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_{\alpha}(-1)}\right]$$

$$=Ln_{\alpha}\left(\left|2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right|\right)-\frac{1}{2}\left({}_0I_x^{\alpha}\right)\left[\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)-1\right]\otimes_{\alpha}\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_{\alpha}(-1)}\right]$$

$$=Ln_{\alpha}\left(\left|2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right|\right)$$

$$-\frac{1}{2}\left({}_0I_x^{\alpha}\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_{\alpha}(-1)}\otimes_{\alpha}\left({}_0D_x^{\alpha}\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]\right]$$

$$+\frac{1}{2}\left({}_0I_x^{\alpha}\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2}\right]^{\otimes_{\alpha}2}+\frac{3}{4}\right]^{\otimes_{\alpha}(-1)}\right]$$

$$=Ln_{\alpha}\left(\left|2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right|\right)-\frac{1}{2}Ln_{\alpha}\left(\left|\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right|\right)$$

$$+\frac{1}{\sqrt{3}}arctan_{\alpha}\left[\frac{1}{\sqrt{3}}\cdot\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\right]-\frac{1}{\sqrt{3}}arctan_{\alpha}\left(\frac{1}{\sqrt{3}}\right). \tag{20}$$

Example 3.2: If $0<\alpha\leq 1$. Find the α-fractional integral

$$\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-3\right)\otimes_{\alpha}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\otimes_{\alpha}\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}2}\right)\right]^{\otimes_{\alpha}(-1)}\right]. \tag{21}$$

Solution Also by undetermined coefficients method, let

$$\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-3\right)\otimes_{\alpha}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\otimes_{\alpha}\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}2}\right)\right]^{\otimes_{\alpha}(-1)}$$

$$=D\cdot\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\right]^{\otimes_{\alpha}(-1)}+E\cdot\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}(-1)}+F\cdot\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}(-2)}. \tag{22}$$

Where D,E,F are constants. Then

$$\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-3=D\cdot\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}2}+E\cdot\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}-1\right)+F\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right). \tag{23}$$

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Hence,

$$\begin{cases} D+E=0, \\ -2D+F=1 \quad , \\ D-E+F=-3. \end{cases} \tag{24}$$

Thus,

$$\begin{cases} D=-1, \\ E=1 \;, \\ F=-1. \end{cases} \tag{25}$$

Therefore,

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-3\right)\otimes_{\alpha}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\otimes_{\alpha}\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}2}\right)\right]^{\otimes_{\alpha}(-1)}\right]$$

$$=\left({}_{0}I_{x}^{\alpha}\right)\left[-\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\right]^{\otimes_{\alpha}(-1)}+\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}(-1)}-\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}(-2)}\right]$$

$$=-\left({}_{0}I_{x}^{\alpha}\right)\left[\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\right]^{\otimes_{\alpha}(-1)}\right]+\left({}_{0}I_{x}^{\alpha}\right)\left[\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right)\right]^{\otimes_{\alpha}(-1)}\right]-\left({}_{0}I_{x}^{\alpha}\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}(-2)}\right]$$

$$=-Ln_{\alpha}\left(\left|\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right|\right)+Ln_{\alpha}\left(\left|\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right|\right)+\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]^{\otimes_{\alpha}(-1)}+1\,. \tag{26}$$

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we study two fractional integral problems. By undetermined coefficients method, we can find the solutions of these two fractional integrals. In fact, our results are generalizations of traditional calculus results. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in fractional calculus and applied mathematics.

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